ABSTRACT

for the Dissertation on the topic “Spectral analysis of self-adjoint differential–algebraic operators with boundary terms” for the degree Doctor of

Philosophy (PhD) in the specialty “8D05401 – Mathematics”

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Differential–algebraic equations with boundary conditions play a key role in modern research in both theoretical and applied mathematics. They are widely used for modeling dynamical systems in which algebraic constraints are present alongside differential relations. One of the central problems in this area is the study of the existence and uniqueness of solutions to such systems under various types of boundary conditions.

Relevance of the study and literature review.

The analysis of systems of differential–algebraic equations with boundary terms in contemporary research is based on a combination of differential and algebraic methods, as well as the application of operator-theoretic techniques in studying their fundamental properties. Classical methods of spectral analysis of differential operators do not always allow one to adequately account for the influence of algebraic and boundary terms on the structure of solutions and the spectrum. The relevance of the topic is therefore determined by the need for a deeper investigation of the properties of differential–algebraic operators with boundary terms, including issues of self-adjointness, the Riesz basis property of systems of eigenfunctions and associated functions, and the analytic properties of the resolvent. These questions are important for the development of operator theory and applications in mathematical physics. The intensive development of the theory of differential–algebraic equations (DAEs) began in the second half of the twentieth century under the influence of problems arising in mechanics and electrical circuit modeling.

The general theory of boundary value problems for ordinary differential equations plays an important role both theoretically and practically. Over time, practical demands led to an expansion of this theory to include the study of boundary value problems for differential–boundary equations. The field of differential equations with boundary conditions, often referred to as differential–boundary equations, has been studied from the early twentieth century to the present, with particular

emphasis on recent advances. In 1911, E. Hilb observed that adjoint problems for differential equations with integral conditions can be reduced to differential boundary problems. In 1959, R. Phillips discovered a similar phenomenon for hyperbolic systems of partial differential equations. Phillips was among the first researchers to work directly with differential–boundary operators, and his studies contributed to a more rigorous understanding of the nature of such equations. He investigated methods related to minimal and maximal operators and their dissipative extensions, which are directly connected with the concept of differential–boundary equations. His work demonstrated the necessity of extending differential operators to differential–boundary operators, showing that in some cases differential operators naturally transform into differential–boundary operators containing both differential parts and boundary operators.

According to the terminology introduced by A. Kroll, differential–boundary equations, in addition to differential expressions, include terms containing boundary values of the unknown function. Boundary value problems constitute an important area of applied mathematics for explicitly given ordinary differential equations, and this statement is even more applicable to differential–algebraic equations. Early focused studies of boundary value problems for differential–algebraic systems were related to issues of regularization and consistency of boundary conditions. Within these works, linear DAEs with classical boundary conditions were analyzed, and conditions ensuring well-posedness were established.

The present dissertation is devoted to the investigation of a number of problems in the theory of differential equations and operators with complex structure that are of significant scientific interest from the standpoint of modern functional analysis. Although many application areas could be mentioned, the examples given already demonstrate the wide applicability of differential–algebraic equations in mathematical modeling and numerical solution of applied problems.

Scientific novelty of the research.

The dissertation presents new results related to the spectral analysis of differential–algebraic operators with boundary terms:

1. A class of differential–algebraic operators is considered in which the differential part is supplemented by algebraic and boundary terms. A formulation of boundary value problems is proposed that allows for the correct definition of the corresponding self-adjoint operators.

2. Conditions for self-adjointness of differential–algebraic operators with boundary terms are investigated within the framework of the considered operator models. It is shown that the choice of boundary conditions significantly influences operator properties.

3. Spectral analysis of self-adjoint differential–algebraic operators with boundary terms is carried out for the studied class of problems.

4. The Riesz basis property of systems of eigenfunctions and associated functions of such operators is studied.

5. A representation of the resolvent of self-adjoint differential–algebraic operators with boundary terms is obtained, and its analytic properties depending on the spectral parameter are analyzed.

Thus, the dissertation systematizes and develops results concerning the spectral analysis of differential–algebraic operators with boundary terms. The spectral-operator problems for differential–algebraic equations with boundary terms considered in Chapters 1–3 have not previously been studied in a systematic way.

Object of the research is a class of differential–algebraic equations with boundary terms and the linear operators generated by these equations in functional spaces.

Subject of the research includes spectral characteristics and operator properties of differential–algebraic equations with boundary terms, including analysis of self-adjointness conditions, description of operator domains, the structure of eigenfunctions and associated functions, and conditions for the Riesz basis property. The research focuses on establishing the interrelation between algebraic constraints, boundary terms, and spectral properties of the corresponding operators.

The main goal of the dissertation is the systematic study of differential–boundary and differential–algebraic equations with algebraic and boundary terms, the formulation and investigation of the corresponding self-adjoint operators, and the analysis of their spectral properties, including the Riesz basis property of eigenfunctions and associated functions and the analytic characteristics of the resolvent.

Research objectives: 1. To study differential–boundary and differential–algebraic equations with algebraic and boundary terms and propose well-posed boundary value formulations.

2. To investigate self-adjointness conditions for the generated operators and their relation to solutions of boundary value problems.

3. To perform spectral analysis of self-adjoint differential–algebraic operators.

4. To study the Riesz basis property of systems of eigenfunctions and associated functions.

5. To obtain a representation of the resolvent and analyze its analytic properties.

Connection with research projects.

The dissertation was carried out during the author's participation in several research projects funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan:

– Project No. AP22783101: "Identification problems for time-dependent coefficients in diffusion equations with a fractional time derivative and nonlocal boundary conditions";

– Project No. AP23485971: "Parabolic equation with a fractional time derivative and a two-dimensional Hamiltonian operator: initial value problem and coefficient identification problems."

Reliability and validity of the results are ensured by the consistent application of methods of functional analysis and spectral theory of linear operators appropriate to the studied class of differential–algebraic equations with boundary terms. The correctness of the conclusions is confirmed by publications in peer-reviewed scientific journals and presentations at international conferences.

Approbation of the work.

The results were presented and discussed at international scientific conferences and seminars, including those held abroad:

-"Традиционная международная апрельская научная конференция в честь Казахстанского дня работников науки" (Алматы, 2023),

-"Уфимская осенняя математическая школа—2024" международная конференция, (Уфа, 2024),

-международная научная конференция студентов и молодых ученых "Фараби Алеми" (Алматы, 2024),

-"Актуальные проблемы математики, физики и информационных технологий в образовании" международная научно-практическая конференция (Ош, 2024),

-Seventh International Conference on Analysis and Applied Mathematics (Turkey, 2024),

-International Conference “Actual problems of analysis, differential equations and algebra” (Astana, 2025),

-15th ISAAC Congress (Astana, 2025),

-“Традиционная международная апрельская научная конференция в честь Казахстанского дня работников науки” (Алматы, 2025),

-International Conference FAIA, (Turkey, 2025),

-Международная научная конференция, “Актуальные вопросы междисциплинарных научных исследований”, (Караганда, 2025),

-International Conference AIP, (Rio-de-Janeiro, 2025),

-“9th Young Scholar Symposium of the East Asia Section of Inverse Problems International Association”, (Kyoto, 2025),

-Trends in Analysis and Differential Equations (Turkestan, 2026).

PUBLICATIONS.

The main results of the study are reflected in five scientific papers and eleven conference abstracts. Two of the papers belong to the Q1 quartile in the Scopus database: the first has a percentile of 81, and the second a percentile of 93.

1.Artykbayeva Zh., Kanguzhin B.“Differential–boundary equations with algebraic terms and associated boundary value problems.” *Carpathian Journal of Mathematics.* 41 (4), (2025): 899–912. **(Scopus percentile – 81, quartile – Q1).**

2. Artykbayeva, Zh., Kanguzhin, B. “Self-Adjoint Differential-Algebraic Operators With Boundary Terms.” *Mathematical Methods in the Applied Sciences*, (2026):1–26. **(Scopus percentile – 93, quartile – Q1).**

3. Artykbayeva, Zh., Kanguzhin, B. “Differential-algebraic equations with boundary terms.” *International Journal of Mathematics and Physics*, *16*(1), (2025): 61-72. **(Scopus percentile – 7, quartile – Q4).**

4. Artykbayeva Zh.N., Mirzakulova A.E., Assilkhan A.A. “Analytical solution of initial value problem for ordinary differential equation with singular perturbation and piecewise constant argument.” *Journal of Mathematics, Mechanics and Computer Science* 122, (2). (2024): 1-13. **(Scopus percentile – 18, quartile – Q4).**

5.Artykbayeva Zh.N., Mirzakulova A.E. “Long-term analysis of solutions with initial jumps in singularly perturbed equations.” *International Journal of Mathematics and Physics* 15, (2). (2024): 110-118. **(Scopus percentile – 7, quartile – Q4).**